Analysis of colored symmetrical patterns (Algebraic Combinatorics)

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Ma. Louise A. N. De Las Peñas, Rene P. Felix and Ma. Veronica P. Quilinguin

Introduction

The study and classification of colored symmetrical patterns continues to be of interest in color symmetry today. A meaningful analysis of colored symmetrical patterns involves the symmetry group G of the uncolored pattern as well as the symmetry group K of the pattern when it is colored. In certain instances, not all elements of G permute the colors and we also consider the subgroup H of elements of G which effect color permutations. This subgroup H contains K as a normal subgroup of elements of H which fix the colors.

A coloring of a symmetrical pattern may be perfect or non-perfect. Perfect colorings occur whenever all the elements of G permute the colors that is, $H = G$; otherwise we have non-perfect colorings.

Perfect colorings have been studied extensively before in [9]. The problem however lies on how to study non-perfect colorings systematically. In the paper "A Framework for Coloring Symmetrical Patterns" by De Las Peñas, Felix and Quilinguin[1], a framework was presented for analyzing both perfect and non-perfect colorings. Moreover, using the framework, all colorings of a symmetrical pattern were determined for which the elements of a given subgroup H of the symmetry group G of the uncolored pattern permute the colors and the elements of a given subgroup K of G fix the colors. In this paper, we shed more light to the study of perfect and non-perfect colorings by giving an alternative proof of this result. For the colorings obtained using the framework, we also find the subgroup H^* consisting of elements of G permuting the colors and the subgroup K^* consisting of elements of G fixing the colors. In [1], the case where the index of H in G is a prime p was considered. In this paper, we present an additional situation where the index of H in G is not prime. Specifically we look at the case where the index of H in G is the smallest composite 4.

Setting for Coloring Symmetrical Patterns

We first explain the setting in which we will color symmetrical patterns. Consider G to be the symmetry group of an uncolored pattern. We start with a fundamental domain for G

and a subset R of this fundamental domain. The set $\{g(R) : g \in G\}$ will be referred to as the G-orbit of R. We assume that the given pattern can be obtained as the G-orbit of some subset R of a fundamental domain for G. Then the assignment $g \longleftrightarrow g(R)$ defines a one-to-one correspondence between the group G and the G-orbit of R. We then can label the set $g(R)$ by g and by giving a color to each $g \in G$, we give a color to each set $g(R)$. This assignment of colors is what we will call a **coloring** of the pattern. Since this results in a partition of G wherein the elements assigned the same color form one set in the partition, a coloring may be treated as simply a partition of the group G or a decomposition of G into non-empty disjoint subsets. Hence, a coloring of a pattern with symmetry group G will be equivalent to a partition of G or a decomposition of G.

We give an example which will illustrate the above concepts. Consider the uncolored pattern in Figure 1.1 which has symmetry group $G = D_6 = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$ where a is a 60°-counterclockwise rotation about the center of the hexagon and b is a reflection in the horizontal line through the center of the hexagon. If R is the triangular region labeled "e" in Figure 1.2, then for each $g \in G$, the triangular region $g(R)$ is labeled "g". Let us partition G into the sets $\{e, a^2, a^4, ab, a^3b, a^5b\}$, and $\{a, a^3, a^5, b, a^2b, a^4b\}$, and assign white and black to the first and second sets respectively. Consequently, we obtain the coloring in Figure 1.3.

In the analysis of a coloring, three groups play a significant role. These groups are:

$G =$ symmetry group of the uncolored pattern

$H =$ subgroup of elements of G which permute the colors

$K =$ subgroup of elements of G which fix the colors

We will refer to H as the subgroup of color transformations and K as the symmetry group of the colored pattern. The groups G, H, K are such that $K \leq H \leq G$. Given a color, its stabilizer in G will lie between H and K. Since H acts on the set C of colors of the pattern, this action induces a homomorphism $f : H \rightarrow A(C)$, where $A(C)$ is the group of permutations of the set C of colors of the pattern. For $h \in H$, $f(h)$ is the permutation of the colors that h induces. An element h is in the kernel of f if and only if $f(h)$ is the identity permutation, that is, h fixes all the colors. Thus the kernel of f is K and the resulting group of color permutations $f(h)$ is isomorphic to H/K. Consequently, K is a normal subgroup of H.

Enumerating Colorings of Symmetrical Patterns

In this part of the paper, we determine all colorings of an uncolored pattern with symmetry group G such that the elements of a given subgroup H of G permute the colors and the elements of a given subgroup K of G fix the colors where $K \leq H \leq N_G(K)$.

The assumptions we are to consider in determining the colorings will be as follows. Let G be a group and H a subgroup of G. Let P be a partition of G. Since a partition of G corresponds to a coloring, we refer to the set P as the set of colors.

Definition 1 *Let G be a group, $H \leq G$, Y a complete set of right coset representatives of H in G, $\bigcup_{i\in I} Y_i$ a decomposition of Y and for each $i \in I$, $J_i \leq H$. Then the coloring or decomposition $G = \bigcup_{i\in I} \bigcup_{h\in H} hJ_iY_i$ or the partition of G, $P = \{hJ_iY_i : i \in I, h \in H\}$ is called a (Y_i, J_i)-H coloring.*

Lemma 2 *A (Y_i, J_i)-H coloring defines an H-invariant partition of G.*

Proof. If $G = \bigcup_{i\in I}^{t} \bigcup_{h\in H} hJ_iY_i$ is a (Y_i, J_i)-H coloring, then it defines an H-invariant partition since for $h' \in H$, $h'G = \bigcup_{i\in I} \bigcup_{h\in H} h'hJ_iY_i = \bigcup_{i\in I} \bigcup_{h\in H} hJ_iY_i$ since premultiplication by $h' \in H$ simply permutes the elements of H. ■

Also, if $K \leq G$ such that $H \leq N_G(K)$ and $K \leq J_i$ for each i, then the elements of K fix each of the sets hJ_iY_i because if $k \in K$ then $khJ_iY_i = hk'J_iY_i = hJ_iY_i$.

Lemma 3 *If $P = \{P_i : i \in I\}$ is a G-invariant partition of the group G, then P is the partition of G consisting of left cosets of some subgroup S of G. This subgroup is the set in the partition containing e.Moreover, the subgroup of elements of G fixing $P = \{P_i : i \in I\}$ is $core_G S$.*

Proof. Let $e \in P_1$ and P_i an arbitrary element of P. If $g \in P_i$, then $g^{-1}g \in g^{-1}P_i$ and $e \in g^{-1}P_i$. Thus, $g^{-1}P_i = P_1$ or $P_i = gP_1$. This means that any element of P, P_i, can be expressed as gP_1 for some $g \in P_i$. If we can show that P_1 is a subgroup of G, then we are done.Now, $g \in G_{P_1}$, the stabilizer of P_1 under left multiplication by elements of $G \Leftrightarrow gP_1 = P_1 \Leftrightarrow g \in P_1$ because $e \in P_1$. Thus, P_1 is the stabilizer of P_1 and P_1 is a subgroup of G.

If we consider $a \in G$, and take any P_i of P where $P_i = gP_1$ for some $g \in P_i$, a fixes $P_i = gP_1$ or $a(gP_1) = g(P_1)$ if and only if $(g^{-1}ag)P_1 = P_1$ so that $g^{-1}ag \in P_1$ and $a \in gP_1g^{-1}$. Thus the subgroup of elements of G fixing the colors in $core_G P_1$. ■

Lemma 4 *Let G be a group, X a non-empty subset of G and K a subgroup of G. Then $kX = X$ for all k in K if and only if X is a union of right cosets of K in G.*

Proof. Assume $kX = X$ for all k in K. Then $X = \bigcup_{x \in X} \{x\}$ is contained in $\bigcup_{x \in X} Kx$. Now $a \in \bigcup_{x \in X} Kx$ implies $a = kx$ for some $k \in K$ and $x \in X$. But $kx \in kX = X$. Therefore $a \in X$. Hence $X = \bigcup_{x \in X} Kx$.

On the other hand, if X is a union of right cosets of K in G, say $X = \bigcup_{g \in A} Kg$, where A is a subset of G, then $kX = \bigcup_{g \in A} kKg = \bigcup_{g \in A} Kg = X$. ■

Theorem 5 *Let G be a group and H a subgroup of G. If P is an H-invariant partition of G, then P corresponds to a decomposition of G in the form $G = \bigcup_{i \in I} \bigcup_{h \in H} hJ_iY_i$ where $\bigcup_{i \in I} Y_i = Y$ is a complete set of right coset representatives of H in G and $J_i \leq H$ for every $i \in I$. If in addition $K \leq H$ and K fixes the elements of P, then $K \leq J_i$ for every $i \in I$.*

Proof. Since P is an H-invariant partition of G, H acts on P by left multiplication. Consider the orbits under the action of H. Let C_i be a color in the ith orbit. Moreover, let J_i be the stabilizer in H of C_i so that $J_iC_i = C_i$. By Lemma 4, C_i is a union of right cosets of J_i, say $C_i = J_iY_i$ where Y_i is a set consisting of one representative for each right coset of J_i contained in C_i. Hence the ith orbit is the set $\{hJ_iY_i : h \in H\}$. So $G = \bigcup_{i \in I} \bigcup_{h \in H} hJ_iY_i$. Note that $\bigcup_{h \in H} hJ_iY_i = (\bigcup_{h \in H} hJ_i)Y_i = HY_i$ so that $G = \bigcup_{i \in I} HY_i$. This implies that $Y = \bigcup_{i \in I} Y_i$ is a complete set of right coset representatives of H in G. If $K \leq H$ and K fixes all elements of P then K fixes C_i. This means that $K \leq J_i$. ■

The above theorem characterizes all partitions of a group G which are invariant under multiplication on the left by elements of a subgroup H of G and whose elements are left fixed by multiplication on the left by elements of a subgroup K of H. It should be mentioned that distinct complete sets of coset representatives of H in G may give rise to the same partition. This situation is addressed in [1].

The Subgroup H* Permuting the Colors and the Subgroup K* Fixing the Colors

Based on the previous theorem, we have determined all colorings of an uncolored pattern with symmetry group G such that the elements of a subgroup H of G permute the colors and the elements of a subgroup K of G fix the colors. The next step is to actually determine for these colorings the subgroup H^* consisting of elements of G permuting the colors and the subgroup K^* of elements of G fixing the colors. At this point, all we can say is that H is contained in H^* and K is contained in K^*.

In the next theorem, given a $(Y_i, J_i)-H$ coloring, we establish the condition for determining when a coloring is perfect, that is, $H^* = G$ and for the special case where $[G : H] = p$ we compute for K^*.

1. The subgroup H* permuting the colors.

Theorem 6 *Let G be a group, $H \leq G$, Y a complete set of right coset representatives of H in G, $\bigcup_{i\in I} Y_i$ a decomposition of Y and for each $i \in I$, $J_i \leq H$. If $G = \bigcup_{i\in I}\bigcup_{h\in H} hJ_iY_i$ is a given (Y_i, J_i)-H coloring, then this coloring is perfect if and only if J_1Y_1 is a subgroup of G and for each i, $i \in I$ there is a $y_i \in Y_i$ such that $y_iJ_1Y_1 = J_iY_i$.*

Proof. Assume the coloring is perfect.. Then each set hJ_iY_i is a left coset of some subgroup of G. This subgroup is the set hJ_iY_i containing e which is J_1Y_1. Therefore, J_1Y_1 is a subgroup of G. Let $y_i \in Y_i$. Then $y_iJ_1Y_1$ is one of the sets hJ_iY_i since the coloring is perfect. This set is J_iY_i since y_i is in this set. Hence $y_iJ_1Y_1 = J_iY_i$. Conversely, assume J_1Y_1 is a group of G and for each $i \in I$ there is a $y_i \in Y_i$ such that $y_iJ_1Y_1 = J_iY_i$. Then $hJ_iY_i = hy_iJ_1Y_1$ is a left coset of the subgroup J_1Y_1. Hence the coloring is perfect since all elements of G permute the left cosets. ■

The next theorem looks at H^* when there is only one orbit of colors under the action of H.

Theorem 7 *Let G be a group, $H \leq G$, Y a complete set of right coset representatives of H in G, $e \in Y$, and $J \leq H$. Let $P = \{hJY : h \in H\}$ be a coloring and H^* the subgroup of G consisting of all elements of G which permute the colors. Let $Y' \subseteq Y$.*

(i) If $H^* = HY'$ then $y'JY = JY$ for all $y' \in Y'$.

(ii) If $y' \in N_G(H)$ and $y'JY = JY$ for all $y' \in Y'$ then $HY' \subseteq H^*$.

Proof. (i) Assume $H^* = HY'$. Since $y' \in Y' \subseteq HY'$, then y' permutes the sets in P and $y'JY$ is the set in P containing y'. This set is JY, hence $y'JY = JY$.

(ii) Assume $y' \in N_G(H)$ and $y'JY = JY$ for all $y' \in Y'$. We show y' permutes the sets in P. Now, $y' \in N_G(H)$ implies that if $h \in H$, there is an $h' \in H$ such that $y'h = h'y'$. Hence $y'hJY = h'y'JY = h'JY$. Thus for all $y' \in Y'$, y' permutes the elements in P. Since H permutes the elements in P, so does HY'. Therefore, $HY' \subseteq H$. ■

In the following corollary, we specialize Theorem 7 to the case where the index of H in G is 4.

Corollary 8 *Let G be a group, $H \leq G$ such that $[G : H] = 4$, $Y = \{y_1 = e, y_2, y_3, y_4\}$ a complete set of right coset representatives of H in G and $J \leq H$. Suppose $P = \{hJY : h \in H\}$ is the given coloring or partition.*

(i) The coloring is perfect if and only if JY is subgroup of G.

(ii) If $H^* \neq G$ then for $i = 2, 3, 4$, $H^* = H \cup Hy_i$ if and only if $H \cup Hy_i$ is a subgroup of G and $y_iJY = JY$. Otherwise $H^* = H$.

Proof. **(i)** This is a consequence of Theorem 6 where $JY = J_1Y_1$.

(ii) This follows from Theorem 7 since H is a normal subgroup of $H \cup Hy_i = H\{e, y_i\}$ when $H \cup Hy_i$ is a subgroup of G. ■

2. The subgroup K* fixing the colors

Now that we have established for certain cases the condition for determining H^*, the subgroup of G consisting of elements of G that permute the colors of the corresponding colored pattern, we can give for these cases the formulas for K^*, the subgroup of G consisting of the elements of G fixing the colors. Notice that K^* is a subgroup of H^* so that in determining K^* we consider only the elements of H^*.

Theorem 9 *Let G be a group, $H \leq G$ such that $[G : H] = p$ where p is prime, Y a complete set of right coset representatives of H in G, $\bigcup_{i=1}^{t} Y_i$ a decomposition of Y and for each $i \in \{1, 2, ..., t\}$, $J_i \leq H$. Suppose $G = \bigcup_{i=1}^{t} \bigcup_{h \in H} hJ_iY_i$ is a given (Y_i, J_i)-H coloring.*

(i) If the coloring is perfect then $K^* = core_G(J_1Y_1)$.

(ii) If the coloring is non-perfect then $K^* = \bigcap_{i \in I} core_H(J_i)$.

Proof. **(i)** If the coloring is perfect, then the given (Y_iJ_i)-H coloring partitions G into the sets of left cosets of J_1Y_1 in G. It follows that $K^* = core_G(J_1Y_1)$.

(ii) On the other hand, if the coloring is non-perfect, then the subgroup H^* permuting the set of colors is H since $[G:H] = p$ and $H \leq H^* \leq G$ implies $H^* = H$ or $H^* = G$. Thus, in determining K^* we consider only elements of H. Let $a \in K^*$. Then $ahJ_iY_i = hJ_iY_i$ for $h \in H$, for all $i \in \{1, 2, ..., t\}$. This implies that if $Y_i = \{y_{i_1}, y_{i_2}, ..., y_{i_r}\}$, then $ahJ_iy_{i_1} \cup ahJ_iy_{i_2} \cup ...ahJ_iy_{i_r} = hJ_iy_{i_1} \cup hJ_iy_{i_2} \cup ...hJ_iy_{i_r}$. Now $a \in H$ so that $ahJ_iy_{i_1} \subseteq Hy_{i_1}$, $ahJ_iy_{i_2} \subseteq Hy_{i_2}$, ..., $ahJ_iy_{i_r} \subseteq Hy_{i_r}$. Since a fixes every color, then a takes $hJ_iy_{i_1}$ to itself in Hy_{i_1}, $hJ_iy_{i_2}$ to itself in Hy_{i_2} and so on. Thus for $a \in K^*$, we have $ahJ_iy_{i_j} = hJ_iy_{i_j}$ for all $i \in \{1, 2, ..., t\}$, $j \in \{1, 2, ..., r\}$. But $ahJ_iy_{i_j} = hJ_iy_{i_j}$ implies $ah \in hJ_i$ or $a \in hJ_ih^{-1}$ for $h \in H$. That is, $a \in \bigcap_{h \in H} hJ_ih^{-1} = core_H J_i$. Therefore $K^* \subseteq core_H(J_i)$. The proof of the inclusion $\bigcap_{i \in I} core_H(J_i) \subseteq K^*$ is straightforward. ■

Theorem 10 *Let G be a group, $J \leq H \leq G$, Y a complete set of right coset representatives of H in G containing e and Y' a subset of Y containing e. Let $P = \{hJY : h \in H\}$ be a partition of G. If $H^* = HY'$ then $K^* = core_{HY'}(JY')$.*

Proof. Since $H^* = HY'$, we limit our attention to H^*. Now $H^* \cap JY = JY'$ and the partition P induces the partition $P^* = \{hJY' : h \in H\}$ on H^*. Since P is H^*-invariant, it follows that P^* is H^*-invariant. Hence the induced coloring P^* is a perfect coloring and JY' is a subgroup of H^*. Correspondingly, the subgroup of H^* fixing all the sets or colors in P^* is $core_{H^*}(JY')$. Consequently, this is also the subgroup of elements of H^* which fix the sets in P, that is, $K^* = core_{HY'}(JY')$. ■

Corollary 11 *Let G be a group, $H \leq G$ such that $[G : H] = 4$, $Y = \{y_1 = e, y_2, y_3, y_4\}$ a complete set of right coset representatives of H in G and $J \leq H$. Suppose $P = \{hJY : h \in H\}$ is the given coloring or partition.*

(i) If the coloring is perfect then $K^* = core_G(JY)$.

(ii) If $H^* = H$ then $K^* = core_H J$.

(iii) If $H^* = H \cup Hy_i$ then $K^* = core_{H \cup Hy_i}(J \cup Jy_i)$ for $i = 2, 3, 4$.

Proof. We obtain (i), (ii) and (iii) by taking $Y' = Y, \{e\}$ and $\{e, y_i\}$ *in* Theorem 10 respectively. ■

We conclude the section by looking at the following examples. An illustration of Corollary 11 is given below.

Example 12 *Let* $G = D_6 = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$ *and* H, K *subgroups of* G *given by* $H = \{e, a^2, a^4\}$, $K = \{e\}$.

Now $G = H \cup Hb \cup Ha \cup Hab$

$G = \{e, a^2, a^4\} \cup \{b, a^2b, a^4b\} \cup \{a, a^3, a^5\} \cup \{ab, a^3b, a^5b\}$ *Among the possible* $Y's$ *are* $\{e, a^3, b, a^3b\}$, $\{e, a, ab, a^4b\}$, $\{e, a^5, a^2b, a^3b\}$ *and* $\{e, a, a^4b, a^5b\}$.

We give some colorings $P = \{hJY : h \in H\}$ *of the hexagon in Figure 2 such that the elements of* H *permute the colors and the elements of* $K = \{e\}$ *fix the colors. In the table below, we give* H^* *and* K^* *as well as the* Y *used for each of the colorings. Note that for all colorings* $J = \{e\}$ *so that* $JY = Y$. *We use the following notation:* w *for white,* s *for striped and* b *for black.*

Coloring	H			Hb			Ha			Hab		
Number	e	a^2	a^4	b	a^2b	a^4b	a	a^3	a^5	ab	a^3b	a^5b
1	w	s	b	w	s	b	b	w	s	b	w	s
2	w	s	b	s	b	w	w	s	b	w	s	b
3	w	s	b	b	w	s	s	b	w	b	w	s
4	w	s	b	s	b	w	w	s	b	s	b	w

Coloring			
Number	Y used	H*	K*
1	$Y = \{e, a^3, b, a^3b\}$	G	$\{e, a^3\}$
2	$Y = \{e, a, ab, a^4b\}$	H	K
3	$Y = \{e, a^5, a^2b, a^3b\}$	$\{e, a^2, a^4, b, a^2b, a^4b\}$	K
4	$Y = \{e, a, a^4b, a^5b\}$	$\{e, a^2, a^4, ab, a^3b, a^5b\}$	K

Example 13 *Consider the colored patterns in Figures 3, 4, 5, 6, 7, 8 which are assumed to repeat over the entire plane. For all the colored patterns, the symmetry group* G *of the patterns with the colors disregarded is a hexagonal plane crystallographic group of type p6m generated by* a, b, x *and* y *where* a *is a* $60°$ *- counterclockwise rotation about the indicated point* P, b *is a reflection in a horizontal line through* P *and* x, y *are translations as indicated. These colored patterns have been obtained by choosing the subgroups* $H = \langle a, x, y \rangle$ *and* $K = \langle a^2, x, y \rangle$ *of*

G. H and K are hexagonal plane crystallographic groups of types p6 *and* p3 *respectively. K is normal in G so that* $G = N_G(K)$. *Observe that the colorings in Figure 7 and Figure 8 are the only non-perfect colorings, that is,* $H^* = H$. *Moreover, for these colorings,* $K^* = K$. *All the other colorings are perfect, so that* $H^* = G$. *For the perfect colorings in Figures 3, 4, 5, 6,* $K^* = H, < a^2, b, x, y >, < a^2, ab, x, y >$ *and K respectively.* $< a^2, b, x, y >$ *and* $< a^2, ab, x, y >$ *are hexagonal plane crystallographic groups of types* p31m *and* p3m1 *respectively..*

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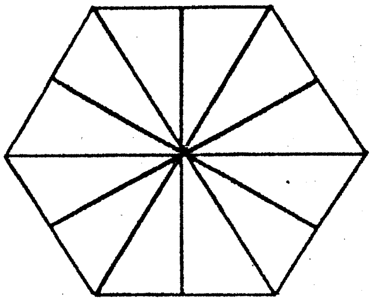

Figure 1.1

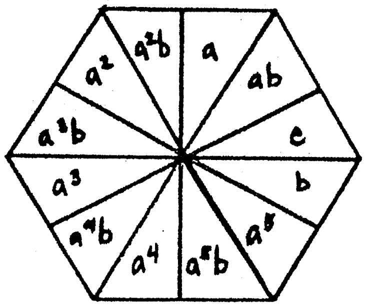

Figure 1.2

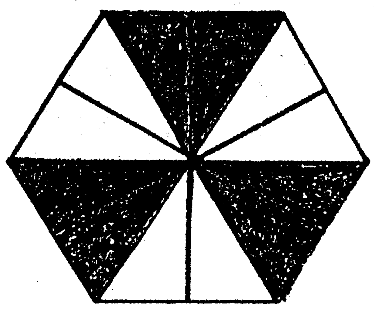

Figure 1.3

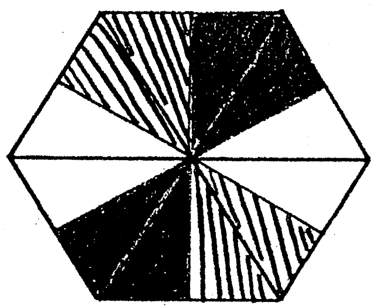

Figure 2.1

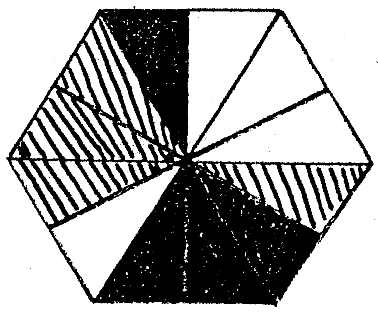

Figure 2..2

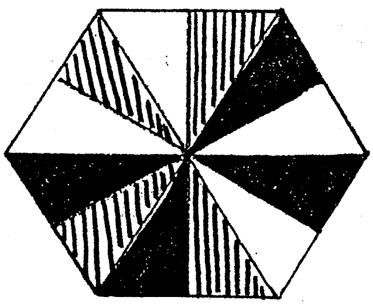

Figure 2.3

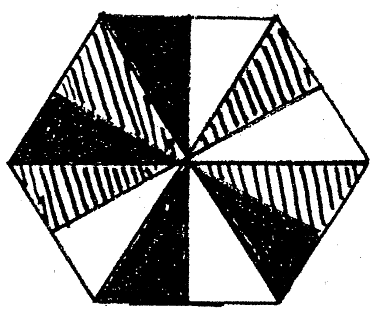

Figure 2.4

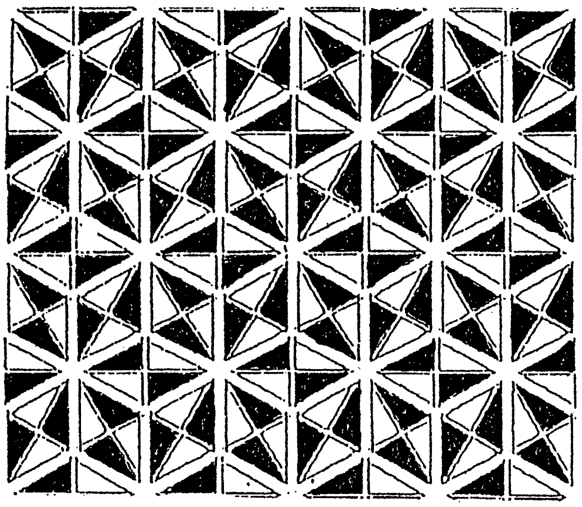

Figure 3

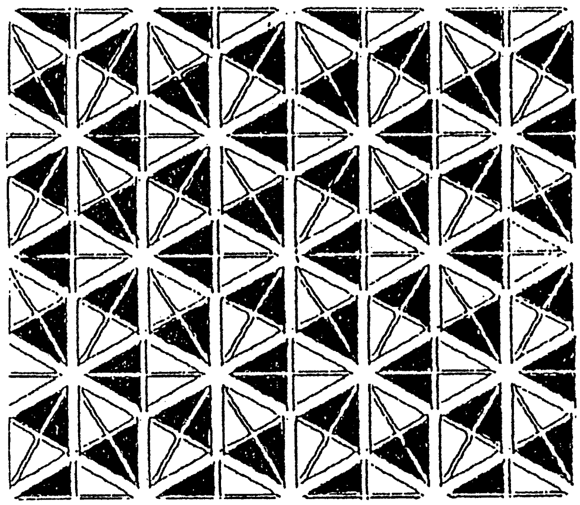

Figure 4

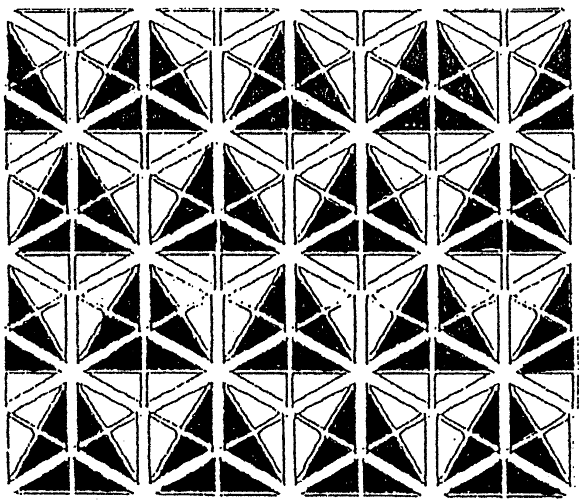

Figure 5

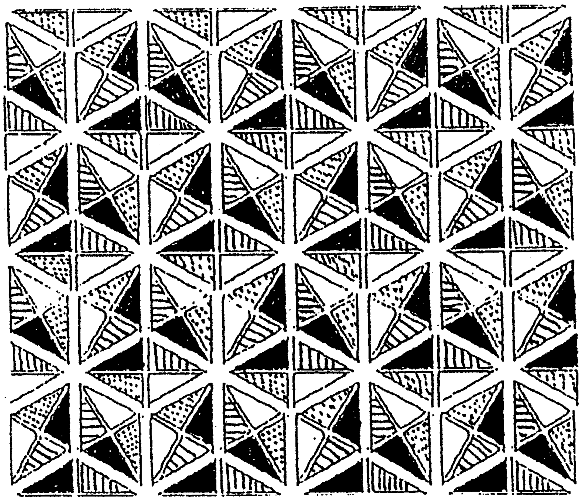

Figure 6

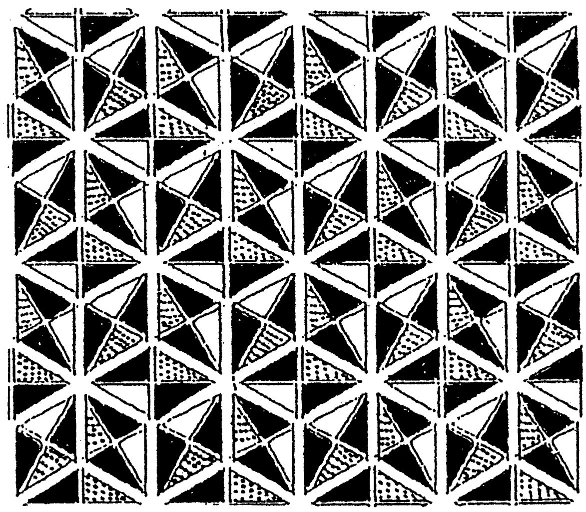

Figure 7

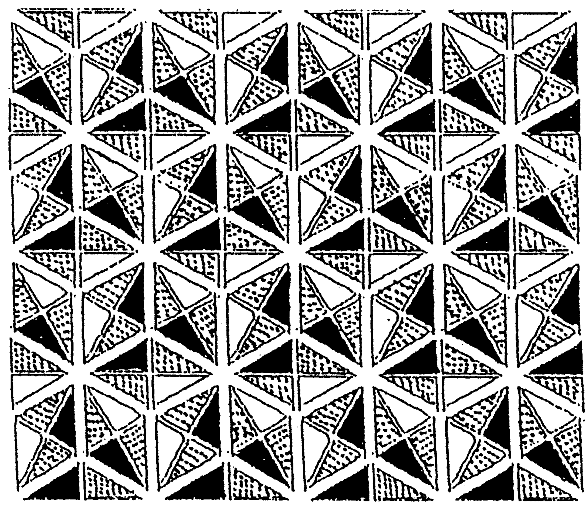

Figure 8